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On global and local mesh refinements by a generalized conforming bisection algorithm

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Abstract: We examine a generalized conforming bisection (GCB-) algorithm which allows both global and local nested refinements of the triangulations without generating hanging nodes. It is based on the notion of mesh density function which prescribes where and how much to refine the mesh. Some regularity properties of generated sequences of refined triangulations are proved. Several numerical tests demonstrate the efficiency of the proposed bisection algorithm. It is also shown how to modify GCB-algorithm in order to generate anisotropic meshes with high aspect ratios.

Keywords: Zlámal's minimum angle condition, finite element method, nested triangulations, conforming longest-edge bisection algorithm, high aspect ratio elements

Mathematical Subject Classification: 65M50, 65N30, 65N50

1 Introduction

Bisection-type algorithms are very convenient for refining simplicial partitions which is needed for many practical problems. They were originally employed for solving nonlinear equations [6, 21, 25]. Various geometric properties of partitions generated by such algorithms were proved in a number of works in the seventies [9, 20, 23, 24, 26, 27]. Later, in the eighties, mainly due to the effort of M.-C. Rivara, bisections became popular also in the FEM (finite element method) community for mesh refinement/adaptation purposes, see some bisection-type algorithms in [15, 16] (and also in later works [18, 19]). Several another variants of the algorithm suitable for standard FEMs were also proposed, analysed and numerically tested in [1, 3, 4, 8, 10, 11, 12, 13, 14, 22]. A practical realization of bisection algorithms is considerably simpler than red, red-green, and green refinements of simplices to provide mesh conformity, especially in the case of local mesh refinements and in three or higher dimensions. Bisections, which always divide areas/volumes of mesh elements only by the factor 2 in any dimension, also allow a more fine local control of the mesh-size.

The guiding rules for mesh refinements/adaptivity often come from a posteriori error estimation which generally delivers estimates in the form of integrals (or elementwise contributions) over the solution domain. Thus, we usually have (or can easily define by some extension procedure) a certain function over a given domain which dictates the actual mesh reconstruction, see e.g. [7]. Its general idea is essentially used in this work, where we propose to modify the standard longest-edge bisection algorithm [10, 15, 16] as follows. We choose for bisection, in general, not the longest edge in a given triangulation, but the edge which has a maximal value of its length multiplied by the value of a mesh density function (defined a priori) at the middle of the edge. Our approach entirely differs from the others, as it does not produce any hanging nodes. Therefore, we do not need any postrefinements of meshes (which can be a rather nontrivial algorithmic task, also requiring considerable additional computational costs, see e.g. [17, p. 2228]) to provide their conformity.

The structure of the paper is as follows. In Section 2 we introduce the concept of mesh density function. In Section 3 we prove that the generated triangulations form families of triangulations. For a special case of a constant mesh density function, we prove that any generated family is strongly regular, i.e., there exists a constant $C > 0$ such that $\operatorname{meas} T \geq Ch^2$ for all triangles $T \in \mathcal{T}_h$ and all triangulations $\mathcal{T}_h \in \mathcal{F}$. In addition, we show in this case that all angles are not less than $\frac{\alpha_0}{2}$, where α_0 is the minimal angle in the initial triangulations (see Sections 4 and 5). This slightly improves the earlier result from [10, p. 1694], where a weaker angle condition was derived. Section 6 is devoted to various numerical tests, and Section 7 to generation of anisotropic meshes.

2 Mesh density function

Let $\Omega \subset \mathbf{R}^2$ be a bounded polygonal domain. By $\mathcal{T}$ we denote a usual conforming (i.e., with no hanging nodes) triangular mesh (called also a triangulation) of $\overline{\Omega}$. Let $\mathcal{E} = \mathcal{E}(\mathcal{T})$ be the set of all edges of all triangles of $\mathcal{T}$.

A sequence $\mathcal{F}$ of triangulations is said to be a *family of triangulations* if for every $\varepsilon > 0$ there exists a triangulation $\mathcal{T} \in \mathcal{F}$ such that $|e| < \varepsilon$ for all edges $e \in \mathcal{E}(\mathcal{T})$, where $|\cdot|$ stands for the Euclidean norm.

Triangular mesh refinements (both – global and local) of $\overline{\Omega}$ can be done by means of a priori given positive *mesh density function* m which is supposed to be Lipschitz continuous over $\overline{\Omega}$, i.e., there exists a constant L such that

$$|m(x) - m(y)| \leq L|x - y|, \quad x, y \in \overline{\Omega}. \tag{1}$$

Such a function should be large over those parts of $\overline{\Omega}$, where we need a very fine mesh and small over those parts of $\overline{\Omega}$, where we do not need a fine mesh (see Section 6). From the positiveness and continuity of m we see that there exists a constant m_0 such that

$$0 < m_0 \leq m(x) \quad \forall x \in \overline{\Omega}. \tag{2}$$

Denote by M_e the midpoint of the edge e and define the *criterion functional*

$$J(e) = |e|\, m(M_e). \tag{3}$$

We shall look for an edge $e^* \in \mathcal{E}$ at which J attains its maximal value, i.e.,

$$J(e^*) = \max_{e \in \mathcal{E}} J(e). \tag{4}$$

In the case we have several such edges we choose anyone of them.

Further, we bisect one or two triangles from $\mathcal{T}$ sharing e^* through the midpoint M_{e^*} (see Figure 1). This refinement strategy will be called the *generalized conforming bisection (GCB-) algorithm*. It is used repeatedly to produce a sequence of conforming nested meshes.

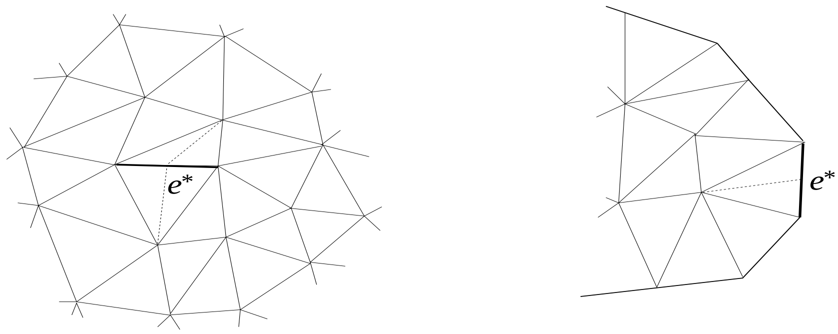

Figure 1: Generalized conforming bisection when e^* is inside Ω and at the boundary $\partial\Omega$. The dotted lines represent the last bisections.

Remark 1 *If $m \equiv 1$ (or if m is constant) then the algorithm is called the conforming longest-edge bisection algorithm. It was analyzed in [10]. The case of constant m is treated in Section 5.*

3 Convergence of GCB algorithm

In [9], Kearfott proved for the longest-edge bisection algorithm (which however produces hanging nodes, in general, see [10, p. 1688]) that the largest diameter of all simplices tends to zero. In Theorem 2 below we prove the same result for the proposed GCB-algorithm. Before that we show that the maximal value of J monotonically tends to zero when the mesh shape is well adapted to the shape of the mesh density function, see (6) below.

Theorem 1 *For each newly generated edge e' after one step of the GCB-algorithm applied to $\mathcal{T}$ we always have*

$$J(e') \leq 0.9\, J(e^*), \tag{5}$$

provided

$$L_T \operatorname{diam} T \leq 0.03 \min_{x \in T} m(x) \qquad \forall T \in \mathcal{T}, \tag{6}$$

where L_T is the minimal possible Lipschitz constant of m on T.

P r o o f . Let e^* be the edge satisfying (4) and (6). Let $T \in \mathcal{T}$ be a triangle which will be bisected. There will be three new edges in T: two halves of e^* and the median to e^*. Let e' be the first (or second) half of e^*. Then from the positiveness and Lipschitz continuity of m we obtain that

$$0.5\, m(M_{e'}) \leq 0.9\, m(M_{e^*}). \tag{7}$$

Indeed, from (1), (6), and (2) we find that

$$m(M_{e'}) \leq m(M_{e^*}) + L_T |M_{e'} - M_{e^*}| = m(M_{e^*}) + \frac{1}{4} L_T |e^*| \leq m(M_{e^*}) + 0.8 \min_{x \in T} m(x) \leq 1.8\, m(M_{e^*}).$$

Hence, (7) holds.

Multiplying (7) by $|e^*| = 2|e'|$, we obtain (5), namely

$$J(e') = |e'| m(M_{e'}) \leq 0.9\, |e^*| m(M_{e^*}) = 0.9\, J(e^*).$$

Now let $e' \subset T$ be the median to e^* and let a, b, c be the lengths of edges of T satisfying

$$a \leq b \leq c. \tag{8}$$

Consider three possible cases:

1) Let $c = |e^*|$. Then by (6), (2), and (3)

$$\frac{3}{8} L_T c^2 \leq \frac{0.09}{8} c \min_{x \in T} m(x) \leq \Big(0.9 - \frac{\sqrt{3}}{2}\Big) J(c). \tag{9}$$

Let $t = |e'|$ be the length of the median on the edge e^* (see Figure 2). Using the Cosine theorem for the both subtriangles of ABC, we have by (8) that $2t^2 = b^2 + a^2 - \frac{c^2}{2} \leq \frac{3c^2}{2}$, i.e.,

$$t \leq \frac{\sqrt{3}}{2} c. \tag{10}$$

Applying (3) and also (10) twice, we find by the Lipschitz continuity of m on T that

$$\begin{aligned} J(e') = t m(M_t) &\leq \frac{\sqrt{3}}{2} c m(M_t) = \frac{\sqrt{3}}{2} c m(M_c) + \frac{\sqrt{3}}{2} c\Big(m(M_t) - m(M_c)\Big) \\ &\leq \frac{\sqrt{3}}{2} J(c) + \frac{\sqrt{3}}{2} L_T c |M_t - M_c| = \frac{\sqrt{3}}{2} J(c) + \frac{\sqrt{3}}{2} L_T c \frac{t}{2} \\ &\leq \frac{\sqrt{3}}{2} J(c) + \frac{\sqrt{3}}{2} L_T c \frac{1}{2} \frac{\sqrt{3}}{2} c = \frac{\sqrt{3}}{2} J(c) + \frac{3}{8} L_T c^2 \leq 0.9\, J(c) = 0.9\, J(e^*), \end{aligned}$$

where the last inequality follows from (9). Thus, (5) holds.

2) Assume now that $b = |e^*|$ and let $u = |e'|$ be the length of the median on e^*. From (6) and (2) we see that

$$\frac{\sqrt{3}}{2} L_T \frac{|e^*|}{4} \leq 0.03 \min_{x \in T} m(x) \leq \Big(0.9 - \frac{\sqrt{3}}{2}\Big) m(M_c).$$

From this, the Lipschitz continuity of m on T, and the fact that $|M_u - M_c| = \frac{b}{4}$ we obtain

$$\frac{\sqrt{3}}{2} m(M_u) \leq \frac{\sqrt{3}}{2} m(M_c) + \frac{\sqrt{3}}{2} L_T \frac{|e^*|}{4} \leq 0.9\, m(M_c).$$

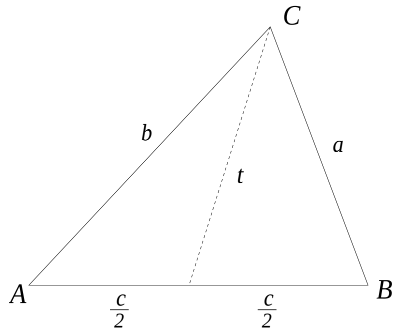

Figure 2: Bisection of a triangle $T \in \mathcal{T}$ for which $c = |e^*|$.

By (8) (and similarly to (10)) we can find that (see Figure 3)

$$u \leq \frac{\sqrt{3}}{2} c. \tag{11}$$

The equality in (11) is attained when the triangle ABC is equilateral as marked in Figure 3. Thus,

$$J(e') = J(u) = um(M_u) \leq \frac{\sqrt{3}}{2} cm(M_u) \leq 0.9cm(M_c) = 0.9\, J(c) \leq 0.9\, J(b) = 0.9\, J(e^*)$$

and (5) holds again.

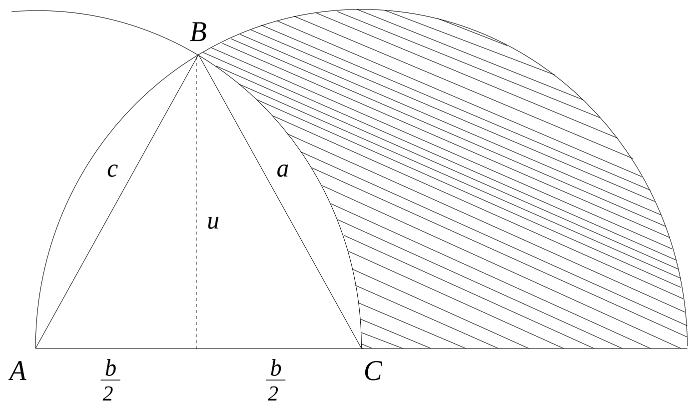

Figure 3: Admissible region for the vertex B. The position of B in its left corner yields the maximal value of the ratio $\frac{u}{c}$ (equal to $\frac{\sqrt{3}}{2}$).

3) Finally, let $a = |e^*|$. We shall distinguish two cases: a) Let $\frac{b}{2} \leq a$. Then $L_T \frac{b}{2} \leq L_T |e^*| = L_T a \leq \frac{1}{9} \min_{x \in T} m(x)$ due to (6). From this, the Lipschitz continuity of m on T and (2) we find further that

$$9m(M_a) \leq 9m(M_c) + 9L_T \frac{b}{2} \leq 9m(M_c) + \min_{x \in T} m(x) \leq 10m(M_c).$$

From above and the inequality $J(c) \leq J(a)$ we obtain that

$$c \leq \frac{m(M_a)}{m(M_c)} a \leq \frac{10}{9} a.$$

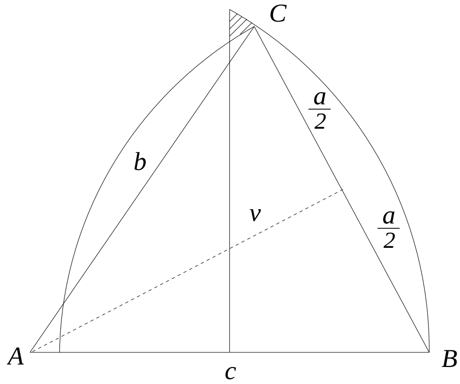

Figure 4: A very small admissible region for the vertex C. The position of C in its right corner yields the maximal value of the ratio $\frac{v}{c}$. The triangle ABC is almost equilateral.

This implies (see Figure 4) that the length v of the median on the edge e^* satisfies

$$v \leq \sqrt{1 - \left(\frac{9}{20}\right)^2} c = \frac{\sqrt{319}}{20} c. \tag{12}$$

The equality on the left-hand side of (12) is attained when the vertex C is in the right corner (where $a = \frac{9}{10}c$ and $b = c$) of the admissible region marked in Figure 4.

From (12), the positiveness and Lipschitz continuity of m on T, (4), and (6) we obtain

$$\begin{aligned} J(e') = J(v) = vm(M_v) &\leq \frac{\sqrt{319}}{20} cm(M_v) \leq \frac{\sqrt{319}}{20} c\Big(m(M_c) + L_T \frac{a}{4}\Big) \\ &\leq \frac{\sqrt{319}}{20} c\Big(m(M_c) + \frac{0.03}{4} \min_{x\in T} m(x)\Big) \leq 0.9\, cm(M_c) = 0.9\, J(c) \\ &\leq 0.9\, J(a) = 0.9\, J(e^*). \end{aligned}$$

b) Now, let $a < \frac{b}{2}$. Then by (6) we have

$$\begin{aligned} J(a) = am(M_a) &< \frac{b}{2} m(M_a) \leq \frac{b}{2}\Big(m(M_b) + L_T \frac{c}{2}\Big) = \frac{1}{2} J(b) + \frac{1}{4} L_T bc \\ &\leq \frac{1}{2} J(b) + \frac{0.03}{4} \min_{x\in T} m(x) b \leq \frac{1}{2} J(b) + \frac{0.03}{4} J(b) < J(b). \end{aligned}$$

However, this contradicts the relation $J(e^*) = J(a) \geq J(b)$. Hence, the case 3b) cannot happen. $\blacksquare$

Theorem 2 *The GCB-algorithm yields a family of nested conforming triangular meshes whose longest edges tend to zero if the initial mesh satisfies condition* (6)*.*

P r o o f . Let T be a triangle that will be bisected. Then all three newly generated edges will be shorter than the longest edge c of T. Therefore, the length of the longest edge of the whole mesh represents a nonincreasing sequence. Thus its limit exists as the bisection proceeds. We prove that this limit is zero.

Let an arbitrary $\varepsilon > 0$ be given and let (6) hold. Consider the edge e^* (to be bisected) from the initial triangulation and let $J(e^*) \geq \varepsilon$. Due to Theorem 1, after bisection of e^*

for all (at most four) newly generated edges e' we have $J(e') \leq 0.9J(e^*)$. Let k be an integer such that

$$0.9^k < \frac{\varepsilon}{J(e^*)}$$

and let n be the number of all edges from the initial triangulation for which $J(e) \geq \varepsilon$. Then we observe that after almost $n4^{k-1}$ bisection steps the functional value $J(e) < \varepsilon$ for all edges in the resulting triangulation. Therefore,

$$|e|m_0 \leq |e|m(M_e) \leq |e^*|m(M_{e^*}) = J(e^*) \to 0.$$

Since m_0 in (2) is positive, we find that also $|e| \to 0$. $\blacksquare$

Remark 2 *The GCB-algorithm can obviously be modified as to allow us, in parallel to refining, also some coarsening of the meshes constructed in those parts of the solution domain where some refinements have previously been made.*

4 Bisection of a single triangle

To prove a nondegeneracy of meshes produced by the conforming longest-edge bisection algorithm we first prove several lemmas. From now on, assume that angles α, β, and γ of an arbitrary given triangle ABC are denoted so that

$$\alpha \leq \beta \leq \gamma, \tag{13}$$

and let again (cf. (8))

$$a \leq b \leq c \tag{14}$$

be lengths of the opposite sides. Now bisect the triangle by the median of length t to the longest side c. Denote the newly generated angles by $\alpha_1, \beta_1, \gamma_1$, and γ_2 as illustrated in Figure 5. If there are two or three sides having the maximum length, then the bisection is not uniquely determined. In this case, we will always bisect that side whose length is denoted by c.

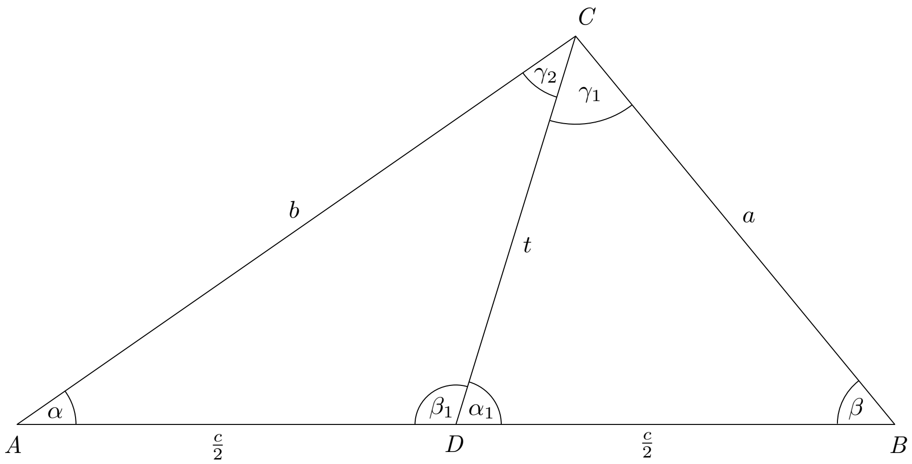

Figure 5: Longest-edge bisection of the triangle ABC.

Lemma 1 *Under the above notation for any triangle we have*

$$\alpha \leq \frac{\pi}{3} \leq \gamma, \quad \beta < \frac{\pi}{2}, \tag{15}$$

$$\alpha_1 \leq \frac{\pi}{2} \leq \beta_1, \tag{16}$$

$$\alpha < \alpha_1, \tag{17}$$

$$\gamma_2 < \frac{\pi}{2}, \quad \gamma_2 \leq \gamma_1, \tag{18}$$

$$\frac{\pi}{6} \leq \gamma_1. \tag{19}$$

P r o o f : Absolute bounds (15) follow from (13) and the equality $\alpha + \beta + \gamma = \pi$.

By the Cosine theorem we see that

$$a^2 = t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\alpha_1,$$

$$b^2 = t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\beta_1.$$

From this and (14) we find that $\cos\alpha_1 \geq \cos\beta_1$. Since $\alpha_1 + \beta_1 = \pi$ and the function cosine is decreasing on the interval $[0, \pi]$, we get (16).

According to

$$\alpha < \alpha + \gamma_2 = \pi - \beta_1 = \alpha_1, \tag{20}$$

we get (17).

From (20) and (16) we immediately see that $\gamma_2 < \pi - \beta_1 \leq \frac{\pi}{2}$, i.e., the first inequality in (18) holds. Using (13), (15), and the Sine theorem, we find that

$$\frac{2\sin\gamma_2}{c} = \frac{\sin\alpha}{t} \leq \frac{\sin\beta}{t} = \frac{2\sin\gamma_1}{c},$$

which yields $\sin\gamma_2 \leq \sin\gamma_1$. From this and the first inequality of (18), we obviously get $\gamma_2 \leq \gamma_1$, because the function sinus is increasing in $[0, \frac{\pi}{2}]$.

Finally, the absolute bound in (19) follows from (15) and (18). ■

Remark 3 Denote vertices of a given triangle ABC as marked in Figure 5. Let D be the midpoint of the longest side AB and let E be such a point that D is the midpoint of the segment CE, i.e., $ACBE$ is a parallelogram. Using the triangle inequality for the triangle ACE and relation (14), we get $2t < a + b \leq 2b$, i.e.,

$$t < b. \tag{21}$$

From (14), (21), and the inequality

$$\frac{c}{2} < \frac{a+b}{2} \leq b,$$

we observe that the triangle ACD (which is never acute due to (16)) will always be bisected in the next step. Its side AC of length b will be halved.

Lemma 2 *Let α be the smallest angle of a nonacute triangle ABC. Bisecting its longest side determines two triangles whose all angles are not less than α.*

P r o o f : The angles α_1, β, β_1, and γ_1 (see Figure 5) can be estimated from below by α due to relations (17), (13), (16), and (18).

Finally we prove that for any nonacute triangle ABC we have

$$\alpha \leq \gamma_2. \tag{22}$$

By the nonacuteness of the triangle and (13) we have $\gamma \geq \frac{\pi}{2}$, and therefore,

$$t \leq \frac{c}{2}. \tag{23}$$

Using the Sine theorem for the triangle ACD, we come to

$$\frac{\sin\alpha}{t} = \frac{2\sin\gamma_2}{c} \leq \frac{\sin\gamma_2}{t},$$

which implies (22) due to (15) and (18). ■

Lemma 3 *For an acute triangle ABC we have*

$$\frac{\alpha}{2} \leq \gamma_2 < \alpha, \tag{24}$$

$$\frac{\pi}{4} < \beta. \tag{25}$$

P r o o f : Let ABC be an arbitrary triangle (not necessarily acute). By (10)

$$t \leq \frac{\sqrt{3}}{2}c\,, \tag{26}$$

where the equality is attained for the equilateral triangle. From (26) and the Sine theorem for the triangle ACD, we get

$$\frac{2\sin\alpha}{\sqrt{3}c} \leq \frac{\sin\alpha}{t} = \frac{2\sin\gamma_2}{c}.$$

Therefore,

$$\sin\alpha \leq \sqrt{3}\sin\gamma_2. \tag{27}$$

Now, assume that ABC is acute. Using (18) and the fact that $\gamma < \frac{\pi}{2}$, we find that

$$\gamma_2 < \frac{\pi}{4}.$$

Consider now two cases:

1) Let $\gamma_2 \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$. Then by (15)

$$\frac{\alpha}{2} \leq \frac{\pi}{6} < \gamma_2,$$

and thus the first inequality of (in ???) (24) holds.

2) Let $\gamma_2 \leq \frac{\pi}{6}$. By (27) and (18),

$$\sin\alpha \leq \sqrt{3}\sin\gamma_2 = 2\cos\frac{\pi}{6}\sin\gamma_2 \leq 2\cos\gamma_2\sin\gamma_2 = \sin 2\gamma_2,$$

which implies that $\alpha \leq 2\gamma_2$ as both angles α and $2\gamma_2$ are from $(0, \frac{\pi}{2}]$, i.e., the first inequality of (24) holds again.

Further, we observe that

$$\frac{c}{2} < t, \tag{28}$$

since the triangle ABC is acute. From this and the Sine theorem for the triangle ACD we find that

$$\frac{2\sin\gamma_2}{c} = \frac{\sin\alpha}{t} < \frac{2\sin\alpha}{c}.$$

Hence, $\gamma_2 < \alpha$ and the second inequality of (24) holds for both cases 1) and 2).

Since $\gamma < \frac{\pi}{2}$, we observe that $\frac{\pi}{2} < \alpha + \beta \leq 2\beta$, which implies (25). ∎

Corollary 1 *Let α be the smallest angle of an acute triangle ABC. Bisecting its longest side determines two triangles whose all angles are not less than $\frac{\alpha}{2}$. The lower bound $\frac{\alpha}{2}$ is attainable while bisecting the equilateral triangle.*

P r o o f : The angles α_1, β, β_1, γ_1, and γ_2 (see Figure 5) can be estimated from below by $\frac{\alpha}{2}$ due to relations (17), (13), (16), (18), and (24). ∎

Before proving that the bisection algorithm guarantees the minimum angle condition, we present two more lemmas.

Lemma 4 *Let ABC be an acute triangle such that $\alpha_1 \geq \beta$ after one bisection. Then the conforming longest-edge bisection algorithm yields inside ABC only four similarity distinct subtriangles whose minimal angle is $\gamma_2 > 27.5°$.*

P r o o f : First, let $\alpha_1 > \beta$. Then from (28) and the Sine theorem for triangle BCD, we see that

$$\frac{c}{2} < t < a. \tag{29}$$

Having in mind Remark 3, we find from (14) and (29) that the sides will be bisected in the following order: c, b, a, t, $\frac{c}{2}$, and $\frac{c}{2}$. During the refinement process we obtain triangulations which consist of at most four different types of subtriangles (see Figure 7). The first type is similar to the original triangle ABC. Two other types are similar to the two triangles produced after the first bisection of ABC (see Figure 5). The remaining fourth type is obtained after the second refinement of ABC. Its angles are γ_1, γ_2, and $\pi - \gamma$. From (18) and the inequality $\gamma < \frac{\pi}{2}$ it follows that the minimal angle of all four types of triangles is γ_2. It attains its minimal value

$$\gamma_2^{\min} = \arctan\sqrt{7} - \arctan(\sqrt{7}/3) \approx 27.89° \tag{30}$$

for $a = \sqrt{2}$ and $b = c = 2$. To see this we can set without loss of generality $A = (-1, 0)$ and $B = (1, 0)$. Then the admissible region G of the vertex C will be the intersection of the following sets (see Figure 6):

- the halfspace $x \geq 0$, since $b \geq a$,
- the halfspace $x \leq \frac{1}{2}$, since $\alpha_1 > \beta$,
- the halfspace $y \geq 0$,
- $x^2 + y^2 > 1$, since ABC is acute, and
- $(x+1)^2 + y^2 \leq 4$, since $c \geq b$.

Taking $C = (x, y)$ from the set $\overline{G}$, we can find that $\alpha_1 = \arctan \frac{y}{x}$ and $\alpha = \arctan \frac{y}{x+1}$. Then $\gamma_2 = \alpha_1 - \alpha = \arctan \frac{y}{x} - \arctan \frac{y}{x+1}$ attains its minimum over $\overline{G}$ at its corner point $C = (\frac{1}{2}, \frac{\sqrt{7}}{2})$, see Figure 6.

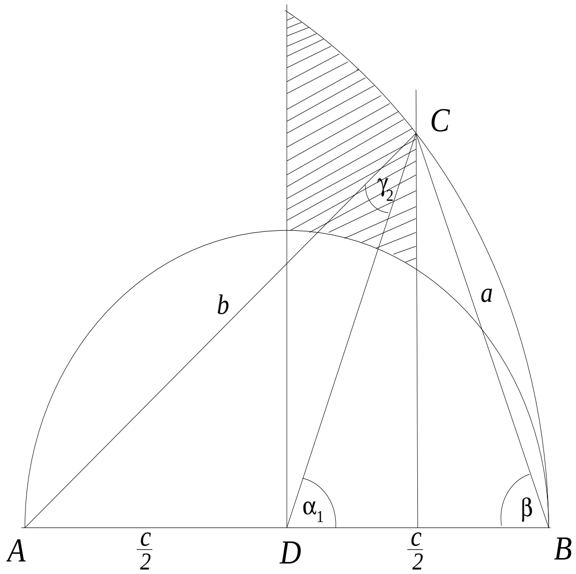

Figure 6: Admissible region for the vertex C. The position of C in its right upper corner yields the minimal value of γ_2.

If $\alpha_1 = \beta$ then BCD is an isosceles triangle and $a = t$. If we first halve t and then a, we obtain a subdivision of BCD that is only a mirror image of the subdivision of BCD, for which a is halved earlier than t. In this case, Lemma 4 holds again. ■

We prove one more lemma keeping the notation of Figure 6, i.e., γ_2 is the angle ACD, where D is the midpoint of the longest side AB.

Lemma 5 *Let ABC be an acute triangle such that $\alpha_1 < \beta$ after one bisection. Let ABC be obtained by the longest-edge bisection of a mother triangle whose minimal angle is α'. Then*

$$\gamma_2 \geq \alpha'.$$

P r o o f : We have the following six possible cases sketched in Figures 8, 9, and 10:

1. Let ABC' be the mother triangle and $|AC'| = 2b$ (see Figure 8). We observe that the considered angle γ_2 is just equal to the angle at C' of the mother triangle ABC', i.e., $\alpha' = \gamma_2$.

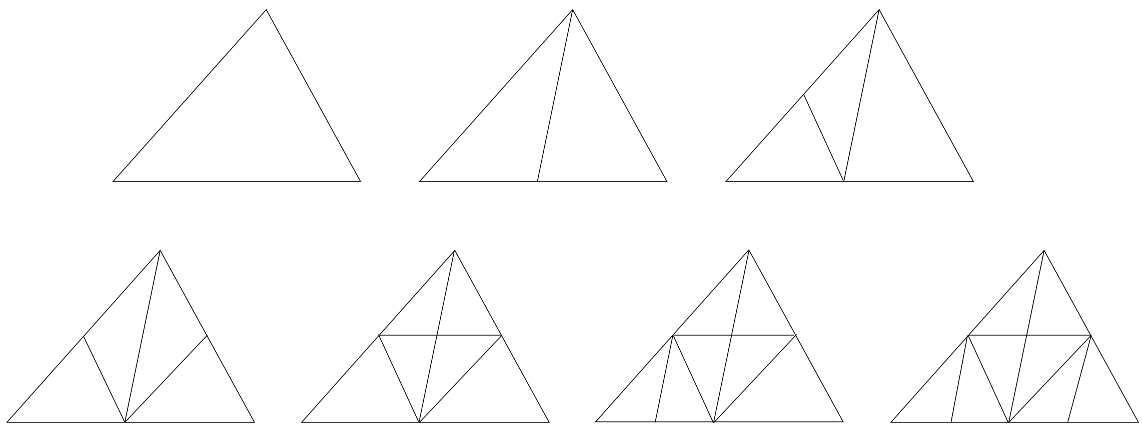

Figure 7: A particular case of the longest-edge bisection algorithm leading to a finite number of similarity distinct subtriangles.

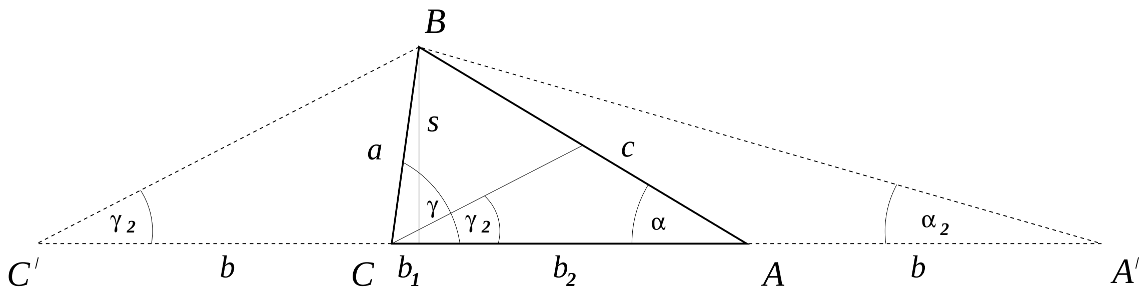

Figure 8: Illustration of cases 1 and 2.

2. Let $A'BC$ be the mother triangle and $|A'C| = 2b$ (see Figure 8). Let s be the length of the altitude on the side AC from B and let

$$b_1 = \frac{s}{\tan\gamma}, \quad b_2 = \frac{s}{\tan\alpha}.$$

Then $b_1 + b_2 = b$, $b_1 \leq b_2$, and therefore,

$$\tan\gamma_2 = \frac{s}{b+b_1} \geq \frac{s}{b+b_2} = \tan\alpha_2,$$

where α_2 is the angle at the vertex A'. Hence,

$$\gamma_2 \geq \alpha_2 = \alpha'. \tag{31}$$

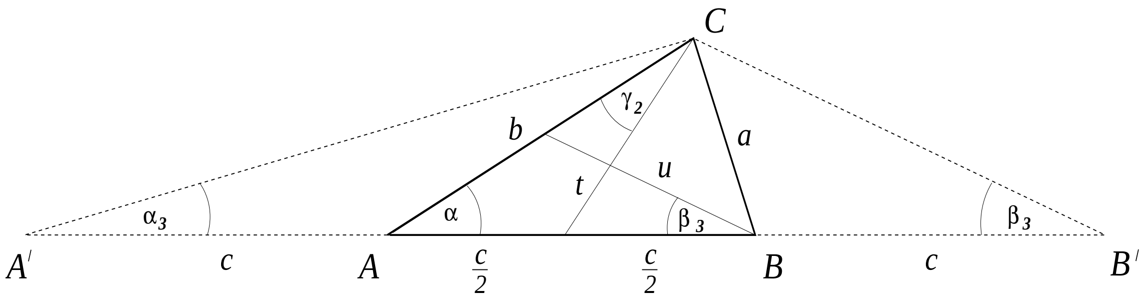

Figure 9: Illustration of cases 3 and 4.

3. Let $AB'C$ be the mother triangle and $|AB'| = 2c$ (see Figure 9). Let β_3 stand for the angle at B', which is acute. Denote by u the length of the median from B to the side AC. Then by the Cosine theorem and (14) we have

$$t^2 = \frac{1}{4}b^2 + \frac{3}{4}b^2 + \frac{1}{4}c^2 - bc\cos\alpha \leq \frac{1}{4}b^2 + c^2 - bc\cos\alpha = u^2,$$

i.e.,

$$t \leq u.$$

From this, the Sine theorem, and (14) we get

$$\frac{2\sin\beta_3}{b} = \frac{\sin\alpha}{u} \leq \frac{\sin\alpha}{t} = \frac{2\sin\gamma_2}{c} \leq \frac{2\sin\gamma_2}{b},$$

which implies that

$$\gamma_2 \geq \beta_3 \geq \alpha'.$$

(In fact, it is easy to find that $\beta_3 = \alpha'$.)

4. Let $A'BC$ be the mother triangle and $|A'B| = 2c$ (see Figure 9). Then similarly to (31), we find that $\beta_3 \geq \alpha_3$. Therefore,

$$\gamma_2 \geq \beta_3 \geq \alpha_3 = \alpha'.$$

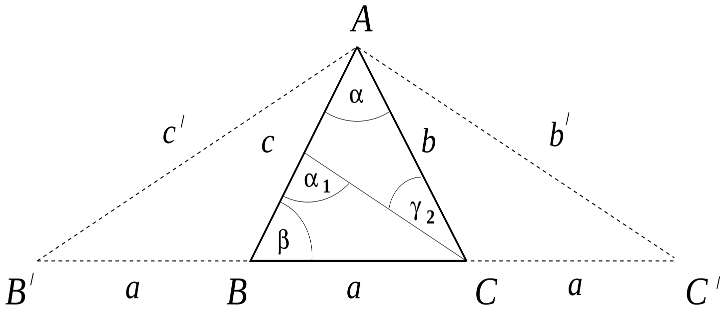

Figure 10: Illustration of cases 5 and 6.

5. Let ABC'' be the mother triangle, $|BC''| = 2a$, and let $b' = |AC''|$ (see Figure 10). We observe that

$$2a \geq b', \tag{32}$$

because the longest side of the mother triangle ABC'' is halved. However, (32) implies that $\alpha_1 \geq \beta$ which contradicts the assumption of the lemma, i.e., the case 5 cannot happen.

6. Let $AB'C$ be the mother triangle, $|B'C| = 2a$, and let $c' = |AB'|$ (see Figure 10). Since the longest side of the mother triangle is halved, we have $2a \geq c'$, i.e., inequality (32) holds as $c' \geq b'$. Thus, we get again a contradiction to the assumption of the lemma and the case 6 cannot happen. ■

5 Regularity results

In this section we shall investigate the case m is constant in $\overline{\Omega}$. In this case the GCB-algorithm reduces to the conforming longest-edge bisection algorithm.

Definition 1 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *regular*, if there exists a constant $C > 0$ such that for all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq C(\operatorname{diam} T)^2.$$

It is well known (see e.g. [5]) that the regularity of $\mathcal{F}$ as defined above is equivalent to the Zlámal's minimum angle condition [28]. Now we will provide a detailed analysis of the validity of this angle condition for the family $\{\mathcal{T}_h\}_{h\to 0}$.

Theorem 3 *Let α_0 be the minimum angle of all triangles from an initial triangulation $\mathcal{T}^0$. Then the conforming longest-edge bisection algorithm yields the following lower bound upon any angle φ of any triangle from any triangulation $\mathcal{T}_h \in \mathcal{F}$*

$$\varphi \geq \hat{\alpha} := \frac{\alpha_0}{2}. \tag{33}$$

P r o o f : 1) Without loss of generality we may investigate each triangle from the initial triangulation $\mathcal{T}^0$ separately. Denoting α_T the minimum angle of a particular triangle $T \in \mathcal{T}^0$, we have

$$\alpha_0 = \min_{T\in\mathcal{T}^0} \alpha_T \leq \frac{\pi}{3}, \tag{34}$$

and thus

$$\hat{\alpha} \le \frac{\pi}{6}. \tag{35}$$

So let an arbitrary triangle $T \in \mathcal{T}^0$ be given. We keep the notation of Figure 5 for T. After the first step of the longest-edge bisection algorithm the minimum angle of the nonacute subtriangle ACD will be $\alpha = \alpha_T$ or γ_2. Hence, by Lemmas 2 and 3, all angles of ACD are not less than $\alpha_T/2 \ge \hat{\alpha}$.

For the subtriangle BCD we have by (17), (13), and (19) that

$$\alpha_1 > \alpha, \quad \beta \ge \alpha, \quad \gamma_1 \ge \frac{\pi}{6},$$

i.e., its minimum angle is greater than or equal to $\min\bigl(\alpha, \frac{\pi}{6}\bigr)$ which is not less than $\alpha_T/2$ due to the first inequality in (15). Thus, we observe that all angles of the both subtriangles ACD and BCD are not less than $\alpha_T/2 \ \ge \ \hat{\alpha}$.

2) Next, we will continue by induction. Consider now an arbitrary triangle T from a triangulation $\mathcal{T}_h$ obtained by the longest-edge algorithm. Let A, B, C be its vertices. Assume that ABC will be bisected in the next step and that it does not belong to the initial triangulation $\mathcal{T}^0$. We will again keep the notation of Figure 5. Further, assume that all angles of all triangles (including the triangle ABC itself) from the considered triangulation $\mathcal{T}_h$ and from all previous triangulations of T are not less than $\hat{\alpha}$, i.e., (33) is valid.

First let ABC be nonacute. Then by Lemma 2, the bisection algorithm does not decrease the value of the minimum angle. This implies that all angles after bisection are not less than $\hat{\alpha}$.

Second assume that ABC is acute. Then by (17), (25), and (19) we come to

$$\alpha_1 > \alpha, \quad \beta > \frac{\pi}{4}, \quad \gamma_1 \ge \frac{\pi}{6}.$$

By the induction hypothesis, $\alpha \ge \hat{\alpha}$. The lower bounds for β and γ_1 are also greater than $\hat{\alpha}$ (cf. (35)). Hence, all angles of the subtriangle BCD are greater than $\hat{\alpha}$.

For the subtriangle ACD we have by the induction hypothesis and (16) that

$$\alpha \ge \hat{\alpha}, \quad \beta_1 \ge \frac{\pi}{2}.$$

So it remains to prove that

$$\gamma_2 \ge \hat{\alpha}. \tag{36}$$

Since ABC is not from the initial triangulation $\mathcal{T}^0$, there exists exactly one mother triangle whose longest-edge bisection produces ABC and which belongs to some previous triangulation. Therefore, the induction hypothesis holds also for the mother triangle. Thus all its angles are greater than or equal to $\hat{\alpha}$.

Now if $\alpha_1 < \beta$ then by Lemma 5 we observe that $\gamma_2 \ge \alpha' \ge \hat{\alpha}$, and thus (36) holds.

Finally, let $\alpha_1 \ge \beta$. Assume to the contrary that (36) does not hold, i.e.,

$$\gamma_2 < \hat{\alpha} = \frac{\alpha_0}{2}. \tag{37}$$

Then by (30), $\alpha_0 > 55°$. Let T^0 be the triangle from the initial triangulation $\mathcal{T}^0$ that contains ABC and let $\alpha^0 \le \beta^0 \le \gamma^0$ be angles of T^0. The upper index 0 will be associated also to the other angles corresponding to the triangle T^0. From (34) we find that

$$\alpha^0 \ge \alpha_0 > 55°. \tag{38}$$

Hence,

$$\gamma^0 = 180^\circ - \alpha^0 - \beta^0 < 180^\circ - 2\alpha^0 < 70^\circ \tag{39}$$

and T^0 is acute. According to (20), (24), (38), and (39),

$$\alpha_1^0 = \alpha^0 + \gamma_2^0 \geq \alpha^0 + \frac{\alpha^0}{2} > 70^\circ > \beta^0.$$

Consequently, T^0 satisfies the assumptions of Lemma 4. Thus, the conforming longest-edge bisection algorithm applied to T^0 yields all angles not less than γ_2^0. Using (24), we get $\gamma_2^0 \geq \alpha^0/2$. Therefore, by (38), all angles during the bisection process will also be not less than $\alpha_0/2$, which contradicts (37). ■

Further, we will prove even a stronger result which shows, in addition, that all produced triangles have approximately the same size for a sufficiently small value of h, even when the initial triangulation contains some triangles of very different sizes.

Definition 2 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *strongly regular*, if there exists a constant $C > 0$ such that for all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq Ch^2.$$

Theorem 4 *The conforming longest-edge bisection algorithm yields a strongly regular family of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$.*

P r o o f : Assume that all sides of all triangles from $\mathcal{T}^0$ were already halved at least one time, and analyse only triangulations produced after these initial refinement steps. Denote any of such triangulations by $\mathcal{T}_h$, where h is the length of the longest side. Let $T \in \mathcal{T}_h$ be that triangle with the shortest side (denoted by a) in the whole triangulation $\mathcal{T}_h$. Since all sides from the initial triangulation were already halved, there exists exactly one mother triangle T' from some previous triangulation such that the bisection of T' in the next step yielded T and the diameter of T' is h'. Then we obtain either $h' = 2a$, or $h' = 2b$, or $h' = 2c$ (cf. Figures 8, 9, and 10), where a, b, and c are the sides of T. Therefore,

$$2c \geq h' \geq h.$$

Further, by the Sine theorem for the triangle T and Theorem 3 we see that

$$a = c\frac{\sin\alpha}{\sin\gamma} \geq c\sin\alpha \geq c\sin\hat{\alpha},$$

and thus,

$$\frac{\sin\hat{\alpha}}{2}h \leq a \leq b \leq c \leq h. \tag{40}$$

From this we get that

$$\operatorname{meas} T = \frac{1}{2}bc\sin\alpha \geq \frac{\sin^3\hat{\alpha}}{8}h^2. \tag{41}$$

Since a was the shortest edge in the whole $\mathcal{T}_h$, formulae similar to (40) and (41) hold also for the other triangles from $\mathcal{T}_h$, which, obviously, proves the theorem. ■

Remark 4 *Assume that the mesh density function is not constant. If h is sufficiently small then the mesh density function is almost constant on each triangle. Therefore, bisections proceed almost like for the case of the conforming longest-edge bisection algorithm.*

6 Numerical tests

In solving partial differential equations by the finite element method, the mesh should be fine at those parts of $\overline{\Omega}$, where we expect some singularities or oscillations of the true solution. This is usually based on a posteriori error estimation, shape of $\overline{\Omega}$, behavior of the coefficients, right-hand side, boundary conditions, etc. An appropriate choice of the mesh density function is presented in several examples below.

Test 1 (boundary layer): Let $\Omega = (-1,1)^2$ and $K = \{(x_1,x_2) \,|\, x_1 = -1\}$. The mesh density function used for iterations $1-499$ is $m_1(x) = 1 + 1/(0.1 + \mathrm{dist}(K,x))$, and for iterations $500-1000$ it is $m_2(x) = 1 + 1/(0.01 + \mathrm{dist}(K,x))$.

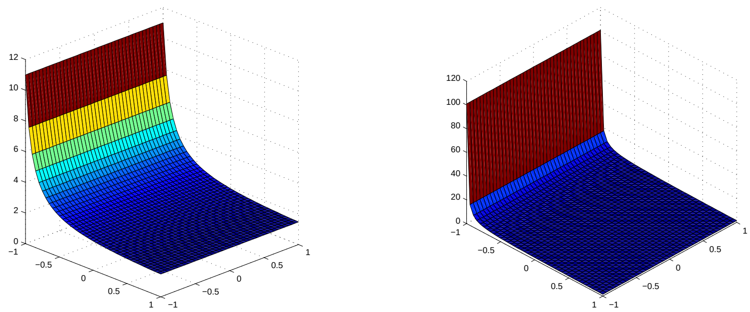

Figure 11: Mesh density functions m_1 (left) and m_2 (right).

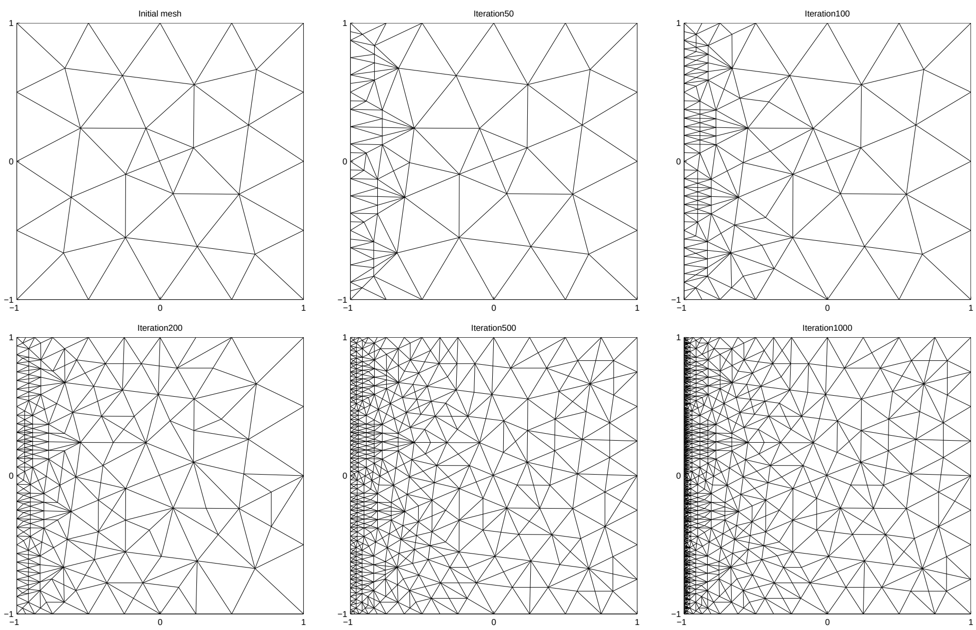

Figure 12: From top left: initial triangulation, triangulations after 50, 100, 200, 500, and 1000 iterations.

Test 2 (interior layer): Let $\Omega = (-1,1)^2$ and $K = \{(x_1, x_2) \,|\, x_1 = x_2\}$. The applied mesh density functions m_1 and m_2 are defined similarly to the previous test.

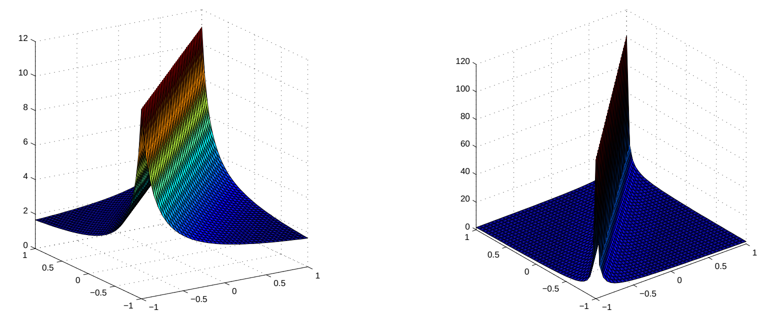

Figure 13: Mesh density functions m_1 (left) and m_2 (right).

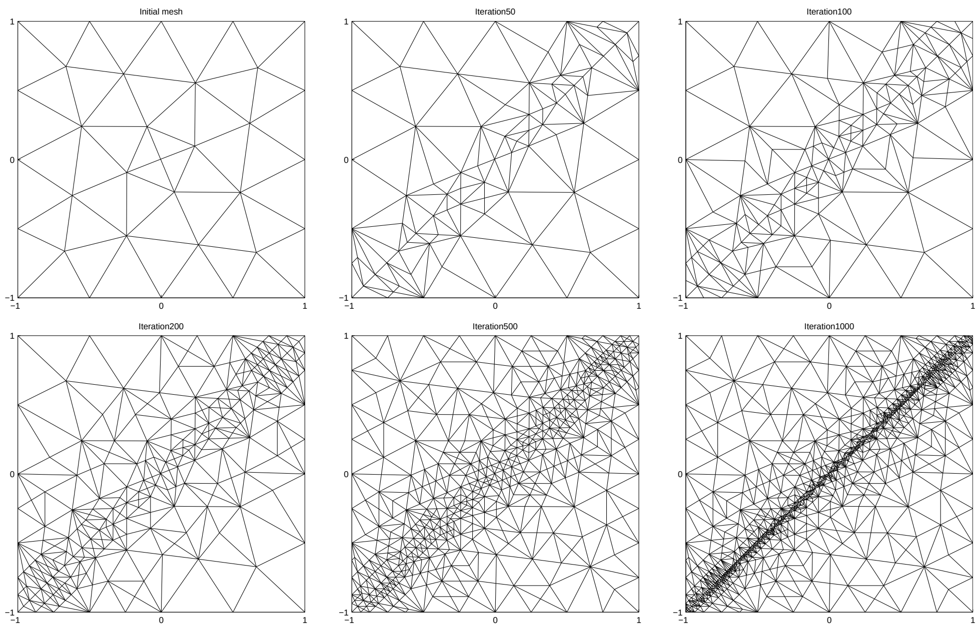

Figure 14: From top left: initial triangulation, triangulations after 50, 100, 200, 500, and 1000 iterations.

Test 3 (two inclusions): Let $\Omega = (-1,1)^2$ and $K_1 = (-0.5,-0.3) \times (-0.1,0.1)$, $K_2 = (0.3,0.5) \times (-0.1,0.1)$. The mesh function used for iterations $1-499$ is $m_3(x) = \sum_i 1/(0.1+\mathrm{dist}(K_i,x))$, and for iterations $500-1000$, $m_4(x) = \sum_i 1/(0.01+\mathrm{dist}(K_i,x))$.

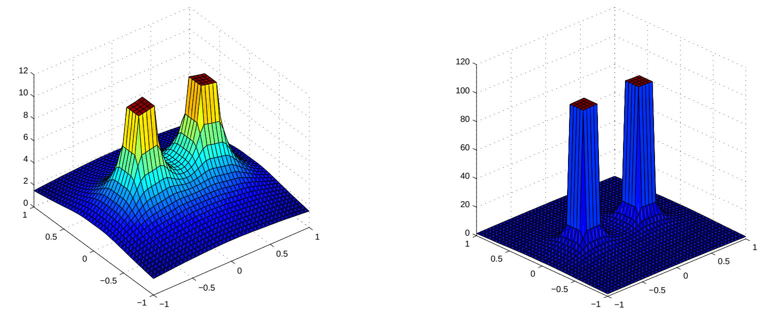

Figure 15: Mesh density functions m_3 (left) and m_4 (right).

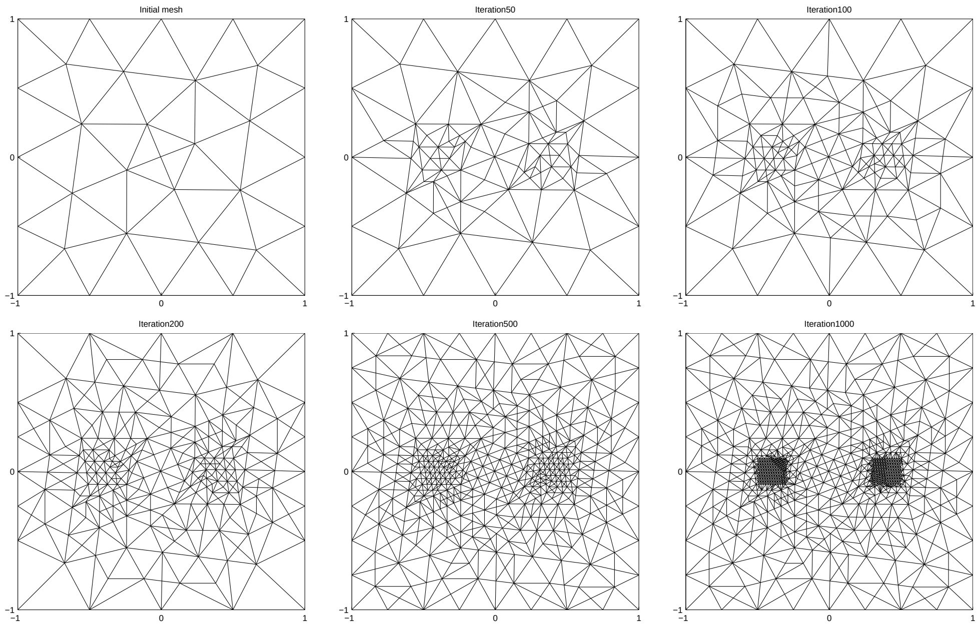

Figure 16: From top left: initial triangulation, triangulations after 50, 100, 200, 500, and 1000 iterations.

Test 4 (L-shaped domain): Let $\Omega = (-1,1)^2 \backslash (0,1)^2$ and $K_1 = (0,0), K_2 = (-1,-1), K_3 = (1,-1), K_4 = (1,0), K_5 = (0,1)$, and $K_6 = (-1,1)$.

The mesh function applied for iterations $1 - 499$ is

$$\sum_i \frac{w_i}{0.1 + \mathrm{dist}(K_i, x)} \tag{42}$$

and for iterations $500 - 1000$

$$\sum_i \frac{w_i}{0.01 + \mathrm{dist}(K_i, x)}. \tag{43}$$

The weight vector is $\{2, 1, 1, 1, 1, 1\}$.

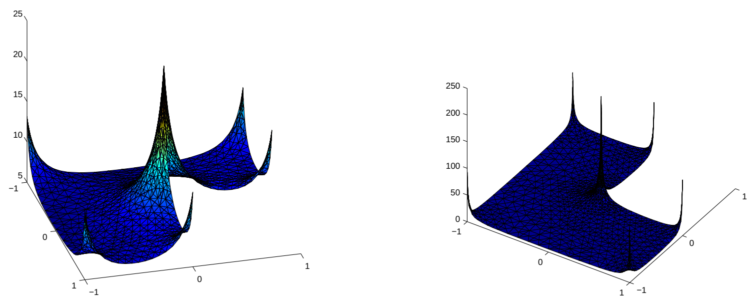

Figure 17: Mesh density functions m_3 (left) and m_4 (right).

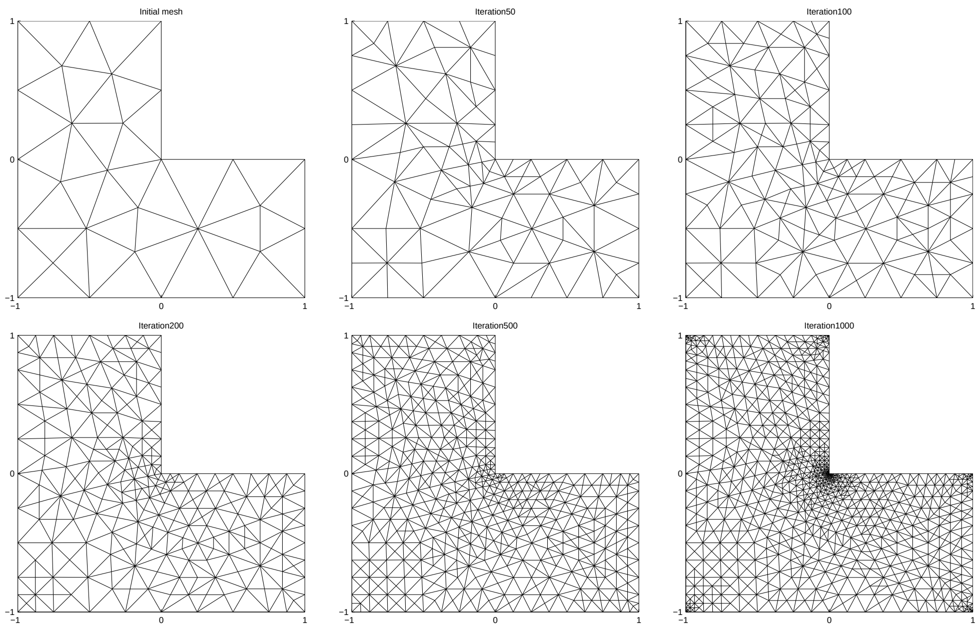

Figure 18: From top left: initial triangulation, triangulations after 50, 100, 200, 500, and 1000 iterations.

7 Anisotropic meshes

We can modify criterion functional (3) so that it will produce elements with high aspect ratio. Consider a vector function $d : \overline{\Omega} \to R^2$ which will determine preferable directions for refinements. Instead of (3) we shall employ the following more sophisticated criterion functional

$$J(e) = |e \cdot d(M_e)| m(M_e), \tag{44}$$

where $\cdot$ stands for the Euclidean scalar product. We can choose d so that it approaches the outward unit normal n near the boundary. In this way we can produce narrow elements near boundary to handle the boundary layers. Functional (44) can also be used to treat anisotropic media, in which material coefficients have different properties in diferent directions. We can also use it for anisotropic triangulations of narrow gaps, thin layers, etc, see [2].

Test 5 (high aspect ratio elements): The domain in $\Omega = (-1, 1)^2$ and $K = \{(x, y)|x = -1\}$. The applied mesh density function for iterations $1 - 499$ is

$$\frac{1}{0.1 + \mathrm{dist}(K, x)} \tag{45}$$

and for iterations $500 - 1000$

$$\frac{1}{0.01 + \mathrm{dist}(K, x)}. \tag{46}$$

The anisotropy weight function $d(x)$ is

$$d(x) = \begin{bmatrix} 1 \\ 0.05 + 0.5\,\mathrm{dist}(K, x) \end{bmatrix} \tag{47}$$

This weight function is applied for all iterations. The anisotropy is visualized by computing bounding box for each triangle and plotting the ratio of x_1 and x_2 width of this box (see Figure 21).

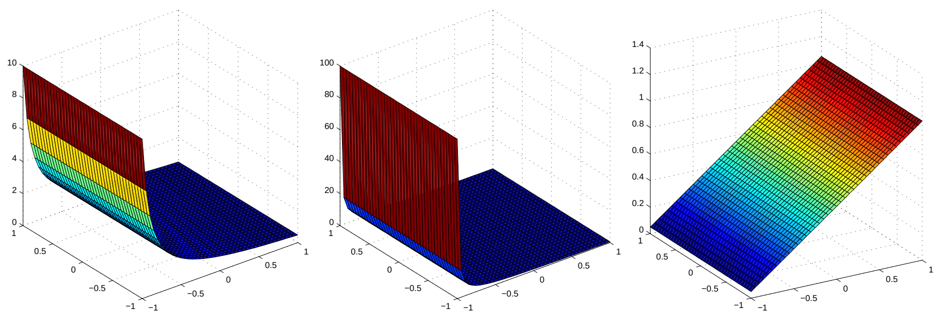

Figure 19: Mesh function for iterations $1 - 499$, mesh function for iterations $500 - 1000$ and anisotropy weight function for iterations $1 - 1000$

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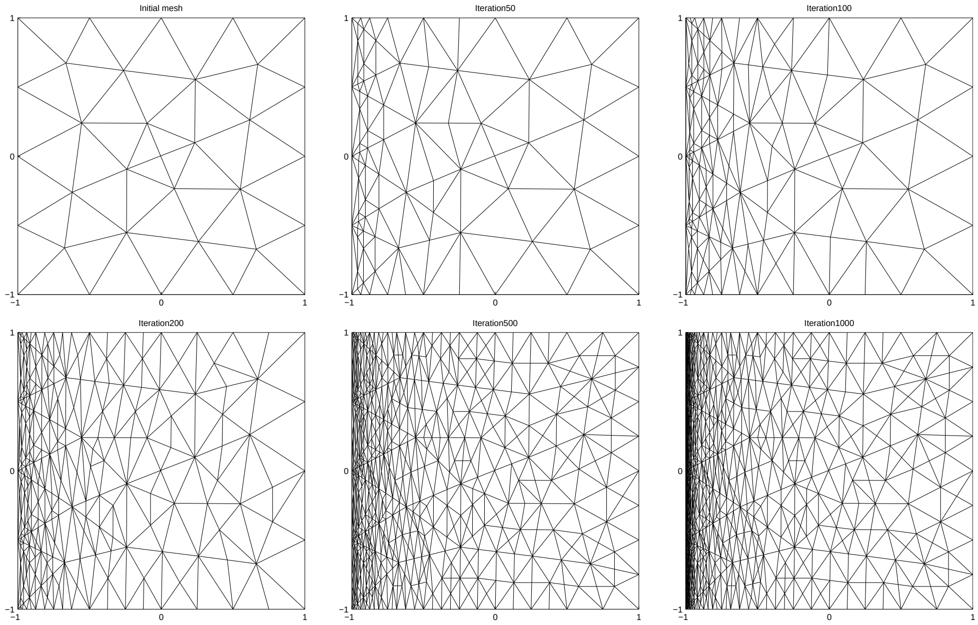

Figure 20: From top left : initial trianglation, triangulations after 50, 100, 200, 500, and 1000 iterations.

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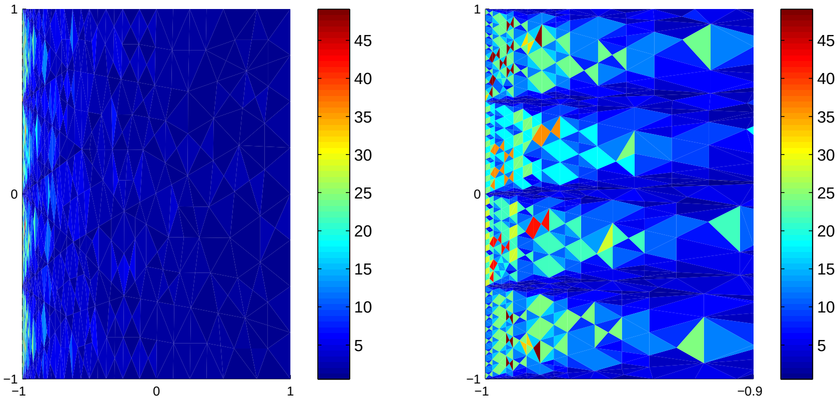

Figure 21: Anisotropy measure for the mesh after 1000 iterations

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